Part 1.3 Special limits

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Exponential Function

In this course we assume that e^x is defined by the series

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

From your first year course you know this converges (absolutely) for all $x \in \mathbb{R}$.

We will further **assume** that $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$. To prove this you need results on when you can multiply series and rearrange the result. Due to lack of time this will have to remain a gap in your knowledge, though details of the general results can be found in Section 3.4, pp 105 - 115, and in particular Corollary on p.114, of the book *Guide to Analysis* by Mary Hart and published by Palgrave Mathematical Guides. For the application of the general results to the specific example of e^x see Theorem 7.4.1. of the same book.

Lemma 1

$$|e^x - 1 - x| \leq |x|^2 \tag{1}$$

for all $|x| \leq 1/2$.

Solution makes use of the fact that if $\lim_{n\to\infty} a_n = \ell$ and the a_n are bounded for all $n \geq 1$, i.e. $|a_n| \leq B$ for some B, then $|\ell| \leq B$. For $x \in \mathbb{R}$ the definition of e^x is

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = \lim_{N\to\infty} \sum_{r=0}^{N} \frac{x^r}{r!}.$$

Thus

$$e^x - 1 - x = \lim_{N\to\infty} \sum_{r=2}^{N} \frac{x^r}{r!} = x^2 \lim_{N\to\infty} \sum_{n=0}^{N-2} \frac{x^n}{(n+2)!} \tag{2}$$

Then, by the triangle inequality, (applicable since we have a **finite** sum),

$$\begin{aligned}
\left| \sum_{n=0}^{N-2} \frac{x^n}{(n+2)!} \right| &\leq \sum_{n=0}^{N-2} \frac{|x|^n}{(n+2)!} \leq \frac{1}{3!} \sum_{n=0}^{N-2} |x|^n \\
&\qquad \text{since } (n+2)! \geq 2! \text{ for all } n \geq 0, \\
&= \frac{1}{2!} \left(\frac{1 - |x|^{N-1}}{1 - |x|} \right),
\end{aligned}$$

on summing the Geometric Series, allowable when $|x| \neq 1$. In fact we have $|x| \leq 1/2 < 1$, which gives the second inequality in

$$\frac{1-|x|^{N+1}}{1-|x|} \leq \frac{1}{1-|x|} \leq \frac{1}{1-1/2} = 2.$$

Hence

$$\left|\sum_{n=0}^{N-2} \frac{x^n}{(n+2)!}\right| \leq 1$$

for all $N \geq 0$. Therefore, since the sequence of partial sums converges, we have

$$\left|\lim_{N\to\infty} \sum_{n=0}^{N-2} \frac{x^n}{(n+2)!}\right| \leq 1.$$

Combined with (2) gives the required result. ■

On the problem sheets you are asked to extend (5) further, including the term $x^3/3!$ in the left hand side.

Theorem 2 *i)*

$$\lim_{x\to 0} e^x = 1.$$

ii)

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Solution i. Open out the result of Lemma 1 as

$$1 + x - |x|^2 < e^x < 1 + x + |x|^2,$$

for $|x| \leq 1/2$. Let $x \to 0$ when $\lim_{x\to 0}\left(1 + x \pm |x|^2\right) = 1$ so, by the Sandwich Rule $\lim_{x\to 0} e^x = 1$.

ii. Divide through (1) by $|x|$ and open up to give

$$1 - |x| < \frac{e^x - 1}{x} < 1 + |x|.$$

The Sandwich Rule will again give the required result. ■

Note "Divide through by $|x|$ and **then** open up"; it will **not** work if you "first open up and then divide by $|x|$".

Aside $\lim_{x\to 0} e^x = 1$ means $e^x - 1 \to 0$ as $x \to 0$. The question then is how fast does $e^x - 1$ tend to 0 as x tends to 0?

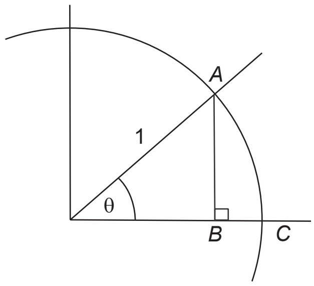
To answer this we look at the ratio, seen in part ii of the theorem. That the limit in part ii exists and is non-zero says that $e^x - 1$ tends to 0 at the same rate as x tends to zero (the value 1 of the limit is irrelevant in this discussion, it only needs to be non-zero).

Trigonometric Functions

Lemma 3 *Using the definitions of* $\sin\theta$ *and* $\cos\theta$ *as the ratio of sides in a right-angled triangle, show that*

$$\lim_{\theta\to 0}\sin\theta = 0 \quad \textit{and} \quad \lim_{\theta\to 0}\cos\theta = 1.$$

Proof First assume $\pi/2 > \theta > 0$. From



we have the lengths of the lines $AB = \sin\theta$, arc length $AC = \theta$ and $BC = 1-\cos\theta$.

Then $AB \le AC$ implies $\sin\theta \le \theta$. Trivially $0 \le \sin\theta$ for $\pi/2 > \theta > 0$, so

$$0 \le \sin\theta \le \theta,$$

for such θ. Let $\theta \to 0+$ to get $\lim_{\theta\to 0+}\sin\theta = 0$ by the one-sided Sandwich Rule.

If $\theta \to 0-$ we use the fact that sin is an *odd* function, i.e. $\sin(-\eta) = -\sin\eta$. Write $\theta = -\eta$ so $\eta \to 0+$. Then

$$\lim_{\theta\to 0-}\sin\theta = \lim_{\eta\to 0+}\sin(-\eta) = -\lim_{\eta\to 0+}\sin\eta = 0,$$

by above.

Since

$$\lim_{\theta\to 0-}\sin\theta = \lim_{\theta\to 0+}\sin\theta = 0$$

we deduce $\lim_{\theta\to 0}\sin\theta = 0$ by an earlier Theorem.

For $\cos\theta$ again start assuming $\pi/2 > \theta > 0$. Then $BC \le AC$ implies $1-\cos\theta \le \theta$, so

$$1-\theta < \cos\theta < 1.$$

Let $\theta \to 0+$ to get $\lim_{\theta\to 0+}\cos\theta = 1$ by the one-sided sandwich rule.

If $\theta \to 0-$ we use the fact that cos is an *even* function, i.e. $\cos(-\eta) = \cos\eta$. Again replacing θ by $-\eta$ we get

$$\lim_{\theta\to 0-} \cos\theta = \lim_{\eta\to 0+} \cos(-\eta) = \lim_{\eta\to 0+} \cos\eta = 1$$

by the result already shown.

Since

$$\lim_{\theta\to 0-} \cos\theta = \lim_{\theta\to 0+} \cos\theta = 1$$

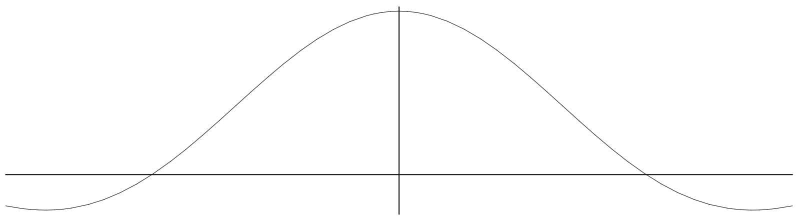
we deduce $\lim_{\theta\to 0} \cos\theta = 1$ by an earlier Theorem. ■

We now come to a fundamental result, with very many applications.

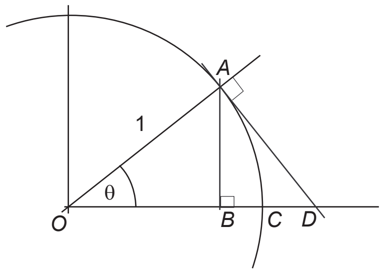
Lemma 4 *Using the definition of* $\sin\theta$ *as the ratio of sides in a right-angled triangle, show that*

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1. \tag{3}$$

Graphically:



Proof Assume $\pi/2 > \theta > 0$. From



we have lengths of lines $OB = \cos\theta$, $AB = \sin\theta$, and $AD = \tan\theta$.

For the areas we have

$$\begin{aligned} OAB &= \frac{1}{2}\sin\theta\cos\theta, \\ OAC &= \frac{1}{2}\theta \times r^2, \\ OAD &= \frac{1}{2}1 \times \tan\theta. \end{aligned}$$

Then the inequality in areas $OAB \le OAB \le OAD$ implies

$$\sin\theta\cos\theta \le \theta \le \tan\theta.$$

This can be rearranged to give

$$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}.$$

Let $\theta \to 0+$ to get

$$\lim_{\theta\to 0+}\frac{\sin\theta}{\theta} = 1,$$

by the one-sided Sandwich Rule.

For $-\pi/2 < \theta < 0$ write $\eta = -\theta > 0$. Then

$$\begin{aligned} \lim_{\theta\to 0-}\frac{\sin\theta}{\theta} &= \lim_{\eta\to 0+}\frac{\sin(-\eta)}{(-\eta)} = \lim_{\eta\to 0+}\frac{-\sin\eta}{(-\eta)} \\ &= \lim_{\eta\to 0+}\frac{\sin\eta}{\eta} = 1 \qquad \text{by above.} \end{aligned}$$

Since

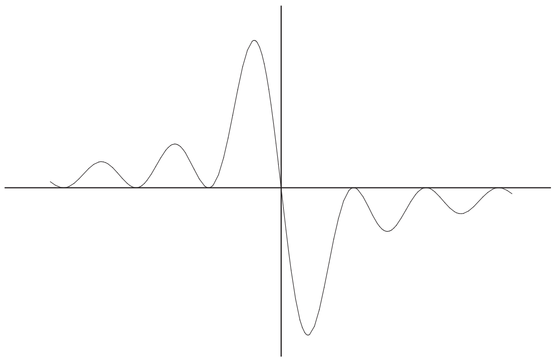
$$\lim_{\theta \to 0+} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0-} \frac{\sin\theta}{\theta} = 1,$$

both one-sided limits exist and are equal and thus the limit exists and equals the common value. ∎

Example 5 *Show that*

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0. \tag{4}$$

Graphically:



Solution For $|\theta| < \pi/2$ we have

$$\begin{aligned} \frac{\cos\theta - 1}{\theta} &= \frac{\cos\theta - 1}{\theta} \times \frac{\cos\theta + 1}{\cos\theta + 1} = \frac{\cos^2\theta - 1}{\theta\,(\cos\theta + 1)} \\ &= -\frac{\sin^2\theta}{\theta\,(\cos\theta + 1)} = -\theta\left(\frac{\sin\theta}{\theta}\right)^2 \frac{1}{\cos\theta + 1}. \end{aligned}$$

Thus by the product and quotient limit rules

$$\begin{aligned} \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} &= -\left(\lim_{\theta \to 0} \theta\right)\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)^2 \frac{1}{\lim_{\theta \to 0} \cos\theta + 1} \\ &= -0 \times 1^2 \times \frac{1}{2} = 0. \end{aligned}$$

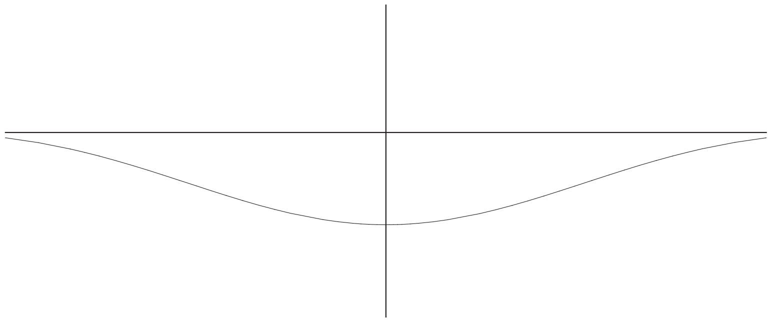
∎

Aside Having a value of 0 for the limit shows that $\cos\theta - 1$ tends to 0 quicker than θ.

Example 6 *Evaluate*

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta^2}.$$

Graphically:



Solution For $|\theta|<\pi/2$ we have

$$\begin{aligned}\frac{\cos\theta-1}{\theta^2} &= \frac{\cos\theta-1}{\theta^2}\times\frac{\cos\theta+1}{\cos\theta+1}=\frac{\cos^2\theta-1}{\theta^2(\cos\theta+1)}\\ &= -\frac{\sin^2\theta}{\theta^2(\cos\theta+1)}=-\left(\frac{\sin\theta}{\theta}\right)^2\frac{1}{\cos\theta+1}.\end{aligned}$$

Thus by the product and quotient limit rules

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta^2}=-\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)^2\frac{1}{\lim_{\theta\to 0}\cos\theta+1}=-\frac{1}{2}.$$

■

Aside Having a non-zero value for the limit shows that $\cos\theta-1$ tends to 0 as fast as θ^2. (Make sure you understand that θ^2 tends to 0 quicker than does θ; squaring numbers less than 1 makes them smaller.)

Important note We have not had to use L'Hôpital's Rule which you may remember from School. But this is how it should be since we haven't yet defined differentiation in this course and so cannot use L'Hôpital's Rule! But even if we had defined differentiation we will see later that we need the two limits above, (3) and (4), to calculate the derivatives of sin and cos. Thus it would be a circular argument to use the derivatives of the trig functions to find, via L'Hôpital's Rule, the values of the limits above which are then used to find the derivatives of the trig functions!